

# Quantum Cloning with Nonlocal Assistance: Complement of Jozsa's Stronger No-Cloning Theorem

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We investigate the cases where a set  $S$  of states  $\{|\psi_i\rangle\}$  cannot be cloned by itself, but is clonable with the help of another system prepared in state  $\hat{\rho}_i$ . When  $S$  is pair-wise nonorthogonal, it is known that one can generate the copy from  $\hat{\rho}_i$  alone, with no interaction with the original system. Here we show that a set containing orthogonal pairs exhibits a property forming a striking contrast; For any such set, there is a choice of  $\hat{\rho}_i$  that enables cloning only when the two systems are interacted in a purely quantum manner that is not achievable via classical communication.

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It is impossible to deterministically make copies of nonorthogonal pure states  $\{|\psi_i\rangle\}_{i=1,\dots,n}$ , as stated in the no-cloning theorem [1, 2]. This property suggests that two quantum copies  $|\psi_i\rangle|\psi_i\rangle$  are more “informative” than one copy  $|\psi_i\rangle$ , and it is natural to ask how much more information the former has than the latter. There have been quantitative approaches to this question, in which one considers quantities such as the optimal success probability in probabilistic cloning protocols [3], or the optimal fidelity in approximate cloning protocols [4, 5, 6, 7, 8, 9, 10, 11]. More recently, Jozsa proposed [12] a qualitative approach by asking what kind of supplementary states  $\{\hat{\rho}_i\}_{i=1,\dots,n}$  is required to make two copies  $|\psi_i\rangle|\psi_i\rangle$  from the original state  $|\psi_i\rangle$ . He found a striking property which he called the stronger no-cloning theorem: For any pair-wise nonorthogonal (PNO) set of original states  $\{|\psi_i\rangle\}_{i=1,\dots,n}$ , whenever two copies  $|\psi_i\rangle|\psi_i\rangle$  are generated with the help of the supplementary state  $\hat{\rho}_i$ , the state  $|\psi_i\rangle$  can be generated from the supplementary state  $\hat{\rho}_i$  alone, independently of the original state, namely,

$$|\psi_i\rangle \otimes \hat{\rho}_i \xrightarrow{\text{CPTP}} |\psi_i\rangle|\psi_i\rangle \implies \hat{\rho}_i \xrightarrow{\text{CPTP}} |\psi_i\rangle, \quad (1)$$

where CPTP stands for a completely positive trace-preserving map. This result implies that the original state is unable to provide even a partial help in the creation of a copy, and hence the cloning process needs no interaction between the original state  $|\psi_i\rangle$  and the supplementary state  $\hat{\rho}_i$ .

While the above theorem only applies to PNO sets, the no-cloning theorem applies to a broader class. We call that a set  $\{|\psi_i\rangle\}$  is “reducible” iff we can divide the set into two nonempty sets  $S_1$  and  $S_2$  such that any state in  $S_1$  is orthogonal to any state in  $S_2$ . Since we can make a projective measurement to distinguish  $S_1$  and  $S_2$  without disturbing the original states, we are allowed

to consider only the irreducible sets in the problem of cloning. When the set of original states is irreducible but not PNO, the cloning is still impossible but the stronger no-cloning theorem no longer applies. Suppose that  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are an orthogonal pair in such a set. As Jozsa pointed out [12], we can take the supplementary information  $\{\hat{\rho}_i\}$  such that  $\hat{\rho}_1 = \hat{\rho}_2$  and any other pair is orthogonal to each other. In this case, the cloning is possible only if we combine the original state and the supplementary state. The required interaction between the two systems is purely classical, namely, if the former system is held by Alice and the latter by Bob, classical communication between them is enough to accomplish the cloning. This example might suggest a plausible interpretation that the part of information held by an orthogonal pair is “classical”, and it can help the creation of a copy by classically communicating with the system holding the supplementary information.

In this paper, we show that there are cases where such an interpretation is not applicable, namely, there are examples of original states  $\{|\psi_i\rangle\}$  and supplementary states  $\{|\phi_i\rangle\}$  that require quantum communication between Alice and Bob to accomplish the cloning. The simplest example is

$$|\psi_1\rangle_A |\phi_1\rangle_B = |0\rangle_A |0\rangle_B, \quad (2)$$

$$|\psi_2\rangle_A |\phi_2\rangle_B = |1\rangle_A |0\rangle_B, \quad (3)$$

$$|\psi_3\rangle_A |\phi_3\rangle_B = 2^{-1}(|0\rangle_A + |1\rangle_A)(|0\rangle_B + |1\rangle_B). \quad (4)$$

It is easy to see that if we apply a controlled-NOT gate between system  $A$  (as control) and system  $B$  (as target), we obtain the cloned state  $|\psi_i\rangle_A |\psi_i\rangle_B$ . On the other hand, as we will prove later, Alice and Bob can never achieve the cloning through local operations and classical communication (LOCC). We further show that this example is not just a special case, but rather represents a general property shared by *all* non-PNO irreducible

sets. We prove that whenever the set of original states  $\{|\psi_i\rangle\}_{i=1,\dots,n}$  is irreducible but not PNO, there always exists a set of supplementary states  $\{|\phi_i\rangle\}_{i=1,\dots,n}$  such that the cloning process requires quantum interaction between the two systems.

Throughout this paper, we assume that Alice holds systems  $A$  and  $A'$ , and Bob holds systems  $B$  and  $B'$ . System  $A$  is secretly prepared in one of the original states  $\{|\psi_i\rangle\}_{i=1,\dots,n}$ , and system  $B$  is prepared in the corresponding supplementary state with the same index  $i$  among the set  $\{|\phi_i\rangle\}_{i=1,\dots,n}$ . When Alice and Bob only communicate classically, difficulty of the cloning tasks depends on the requirement of who should possess the final copies. Since our aim here is to show the impossibility of the task, we adopt the easiest task in which we place no restriction on the locations of the copies, as long as they are known after the protocol. More precisely, we require that the task produces a classical outcome  $XY$  which takes one of the three values  $AA', AB, BB'$ , and the copies are produced accordingly as

$$|\psi_i\rangle_A |\phi_i\rangle_B \xrightarrow{\text{LOCC}} |\psi_i\rangle_X |\psi_i\rangle_Y \quad (i = 1, \dots, n). \quad (5)$$

We consider the following cases with three states:

$$|\psi_1\rangle = |0\rangle, \quad (6)$$

$$|\psi_2\rangle = |1\rangle, \quad (7)$$

$$|\psi_3\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle, \quad (8)$$

where  $\{|i\rangle\}$  is an orthonormal basis, and  $\alpha_0, \alpha_1, \alpha_2$  are real nonnegative numbers satisfying  $\alpha_0 > 0, \alpha_1 > 0$ , and  $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 1$ . Note that this example essentially covers all non-PNO irreducible sets of three states. For the supplementary states, we assume

$$|\phi_1\rangle = |0\rangle, \quad (9)$$

$$|\phi_2\rangle = \left( \alpha_0 \alpha_1 + \sqrt{(1 - \alpha_0^2)(1 - \alpha_1^2)} \right) |0\rangle + \left( \sqrt{1 - \alpha_0^2} \alpha_1 - \alpha_0 \sqrt{1 - \alpha_1^2} \right) |1\rangle, \quad (10)$$

$$|\phi_3\rangle = \alpha_0|0\rangle + \sqrt{1 - \alpha_0^2}|1\rangle. \quad (11)$$

Note that the case with  $\alpha_0 = \alpha_1 = 1/\sqrt{2}$  corresponds to the simple example of Eqs. (2)-(4).

The states  $\{|\phi_i\rangle\}_{i=1,2,3}$  have been chosen such that  $\langle\psi_i|\psi_j\rangle\langle\phi_i|\phi_j\rangle = \langle\psi_i|\psi_j\rangle^2$  for all  $i$  and  $j$ . This relation assures that we can achieve the cloning task by a global operation, namely, there is a unitary  $\hat{U}$  such that  $\hat{U}|\psi_i\rangle_A |\phi_i\rangle_B = |\psi_i\rangle_A |\psi_i\rangle_B$  ( $i = 1, 2, 3$ ). In fact, we can explicitly write down  $\hat{U}$  as follows. Let  $\mathcal{H}_{\text{in}}$  be the subspace spanned by  $\{|\psi_i\rangle_A |\phi_i\rangle_B\}_{i=1,2,3}$ , and  $\mathcal{H}_{\text{out}}^{XY}$  be the one spanned by  $\{|\psi_i\rangle_X |\psi_i\rangle_Y\}_{i=1,2,3}$ . We construct an orthonormal basis of  $\mathcal{H}_{\text{in}}$  by Gram-Schmidt orthogonal-

ization:

$$\begin{aligned} |v_1\rangle_{AB} &:= |0\rangle_A |0\rangle_B, \quad |v_2\rangle_{AB} := |1\rangle_A |\phi_2\rangle_B, \\ |v_3\rangle_{AB} &:= (1 - \alpha_0^4 - \alpha_1^4)^{-1/2} (|\psi_3\rangle_A |\phi_3\rangle_B \\ &\quad - \alpha_0^2 |v_1\rangle_{AB} - \alpha_1^2 |v_2\rangle_{AB}), \end{aligned} \quad (12)$$

and similarly for  $\mathcal{H}_{\text{out}}^{XY}$  as

$$\begin{aligned} |w_1\rangle_{XY} &:= |0\rangle_X |0\rangle_Y, \quad |w_2\rangle_{XY} := |1\rangle_X |1\rangle_Y, \\ |w_3\rangle_{XY} &:= (1 - \alpha_0^4 - \alpha_1^4)^{-1/2} (|\psi_3\rangle_X |\psi_3\rangle_Y \\ &\quad - \alpha_0^2 |w_1\rangle_{XY} - \alpha_1^2 |w_2\rangle_{XY}). \end{aligned} \quad (13)$$

Then,  $\hat{U}$  is simply written as  $\sum_{i=1}^3 |w_i\rangle_{ABAB} \langle v_i|$ .

Now we prove a lemma stating that Alice and Bob have to do a global quantum operation to achieve the cloning:

*Lemma 1* — For the states of Eq. (6)-(11), Alice and Bob can never achieve the cloning task of Eq. (5) over LOCC.

The first step of the proof is to see what happens if Alice and Bob conduct the same cloning protocol with an initial state different from  $|\psi_i\rangle_A |\phi_i\rangle_B$ . Since  $\langle\psi_1|\psi_3\rangle\langle\phi_1|\phi_3\rangle = \alpha_0^2 > 0$  and  $\langle\psi_2|\psi_3\rangle\langle\phi_2|\phi_3\rangle = \alpha_1^2 > 0$ , the set  $\{|\psi_i\rangle|\phi_i\rangle\}_{i=1,2,3}$  is irreducible. This property allows us to determine the output state for a general input state in  $\mathcal{H}_{\text{in}}$ , in the following way. Suppose that after the cloning task of Eq. (5), we swap the states of systems  $X$  and  $A$  (if  $X \neq A$ ), and swap those of  $Y$  and  $B$  (if  $Y \neq B$ ). We further apply the unitary operation  $\hat{U}^{-1}$  to systems  $AB$ . It is easy to see that the whole process does not alter the state of  $AB$  when the initial state is one of the three states  $\{|\psi_i\rangle_A |\phi_i\rangle_B\}_{i=1,2,3}$ . The property of such a disturbance-free process is generally studied in [13, 14]. In the present terminology, the result is stated as follows. If the set of pure states  $S := \{|\chi_i\rangle\}_{i=1,\dots,n}$  preserved in a process is irreducible, the process just leaves the subspace  $\mathcal{H}$  spanned by  $S$  as it is, namely, (a) any state  $|\chi\rangle$  in  $\mathcal{H}$  is preserved in the process, and (b) no information about the identity of the initial state  $|\chi\rangle$  is revealed in the process. Applying this result in the present case, we see that (a') for any initial state  $|\chi\rangle_{AB}$  in  $\mathcal{H}_{\text{in}}$ , the state of systems  $XY$  after the cloning task is  $\hat{U}|\chi\rangle_{AB}$  followed by swaps  $X \leftrightarrow A$  and  $Y \leftrightarrow B$ , and that (b') the probability  $\gamma_{XY}$  of the outcome  $XY$  is independent of the initial state. If we start from a general state  $\sum_{i=1}^3 |v_i\rangle_{AB} |\mu_i\rangle_R$  over  $AB$  and an auxiliary system  $R$ , we should obtain the state  $\sum_{i=1}^3 |w_i\rangle_{XY} |\mu_i\rangle_R$  after the cloning task, and  $XY$  is determined by a probability distribution  $\gamma_{XY}$ , which is fixed for the cloning task and independent of the initial state.

In what follows, we calculate the degree of entanglement of the output states when we try various input states over  $ABR$ . Using the fact that the entanglement never increases over LOCC, we derive bounds on the probabilities  $\gamma_{XY}$ , and show that the condition

$$\gamma_{AA'} + \gamma_{AB} + \gamma_{BB'} = 1 \quad (14)$$

can never be satisfied.

As a measure of entanglement, we use the entanglement monotone for a bipartite state  $|\Omega\rangle_{AB}$  defined by [15]

$$E_{A;B}^{(l)}(\Omega) := 1 - \sum_{i=1}^{l-1} \lambda^{(i)}, \quad (15)$$

where  $\lambda^{(1)}, \lambda^{(2)}, \dots$  is the eigenvalues of  $\text{Tr}_A[|\Omega\rangle_{AB}\langle\Omega|]$  in the decreasing order, and  $l = 2, 3, \dots$ . For each  $l$ , the value of  $E_{A;B}^{(l)}$  never increases on average by LOCC. More precisely, a transformation of a state  $|\Omega^{\text{in}}\rangle_{AB}$  into  $|\Omega_k^{\text{out}}\rangle_{AB}$  with probability  $\gamma_k$  can be done by LOCC iff

$$E_{A;B}^{(l)}(\Omega^{\text{in}}) \geq \sum_k \gamma_k E_{A;B}^{(l)}(\Omega_k^{\text{out}}) \quad (16)$$

for any  $l$  [15].

First we consider the case where system  $R(= A'')$  is held by Alice and try the following state as an input to the cloning process:

$$|\Phi^{\text{in}}\rangle_{ABA''} = \frac{1}{\sqrt{3}} \sum_{i=1}^3 |v_i\rangle_{AB} |i\rangle_{A''}, \quad (17)$$

where  $\{|i\rangle_{A''}\}$  are orthonormal. Then, the process should produce the state

$$|\Phi_{BB'}^{\text{out}}\rangle_{BB'A''} = \frac{1}{\sqrt{3}} \sum_{i=1}^3 |w_i\rangle_{BB'} |i\rangle_{A''} \quad (18)$$

with probability  $\gamma_{BB'}$ . Since all  $\{|\phi_i\rangle_B\}_{i=1,2,3}$  can be expanded by  $|0\rangle_B$  and  $|1\rangle_B$ ,  $E_{A''A;B}^{(3)}(\Phi^{\text{in}}) = 0$ . On the other hand,  $E_{A''A;B}^{(3)}(\Phi_{BB'}^{\text{out}}) = 1/3$ . From Eq. (16), we see that

$$\gamma_{BB'} = 0 \quad (19)$$

for any LOCC cloning process.

Next we consider another case with an input state

$$|\Psi^{\text{in}}\rangle_{ABB''} = \frac{1}{2} \sum_{i=1}^2 |v_i\rangle_{AB} |0\rangle_{B''} + \frac{1}{\sqrt{2}} |v_3\rangle_{AB} |1\rangle_{B''}, \quad (20)$$

where system  $R(= B'')$  is held by Bob. Then the output state of the cloning process is

$$|\Psi_{XY}^{\text{out}}\rangle_{XYB''} = \frac{1}{2} \sum_{i=1}^2 |w_i\rangle_{XY} |0\rangle_{B''} + \frac{1}{\sqrt{2}} |w_3\rangle_{XY} |1\rangle_{B''} \quad (21)$$

with probability  $\gamma_{XY}$ . As in Fig. 1, numerical calculation shows that, for  $\alpha_0 > 0$  and  $\alpha_1 > 0$ ,

$$E_{A;BB''}^{(2)}(\Psi^{\text{in}}) < E_{A;BB''}^{(2)}(\Psi_{AB}^{\text{out}}), \quad (22)$$

$$E_{A;BB''}^{(2)}(\Psi^{\text{in}}) < E_{AA';B''}^{(2)}(\Psi_{AA'}^{\text{out}}). \quad (23)$$

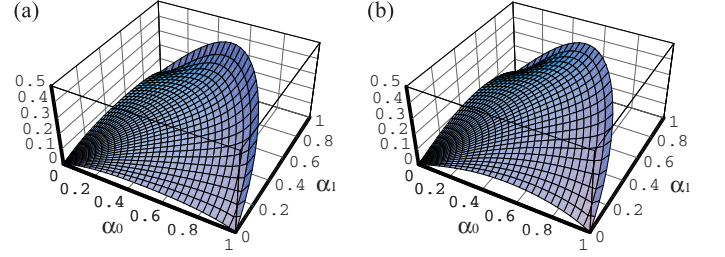


FIG. 1: (a)  $E_{A;BB''}^{(2)}(\Psi_{AB}^{\text{out}}) - E_{A;BB''}^{(2)}(\Psi^{\text{in}})$  and (b)  $E_{AA';B''}^{(2)}(\Psi_{AA'}^{\text{out}}) - E_{A;BB''}^{(2)}(\Psi^{\text{in}})$  as a function of  $(\alpha_0, \alpha_1)$ .

We can also prove these inequalities analytically as follows. Let us define marginal density operators for system  $A$  as  $\hat{\rho}^{\text{in}} := \text{Tr}_{BB''}[|\Psi^{\text{in}}\rangle_{ABB''}\langle\Psi^{\text{in}}|]$  and  $\hat{\rho}^{\text{out}} := \text{Tr}_{BB''}[|\Psi_{AB}^{\text{out}}\rangle_{ABB''}\langle\Psi_{AB}^{\text{out}}|]$ . Eq. (22) is equivalent to the condition for the operator norms,  $\|\hat{\rho}^{\text{in}}\| > \|\hat{\rho}^{\text{out}}\|$ . The difference between the two operators takes a very simple form,  $\hat{\rho}^{\text{in}} - \hat{\rho}^{\text{out}} = \kappa(|0\rangle\langle 1| + |1\rangle\langle 0|)$  with  $\kappa > 0$ . When  $\alpha_2 = 0$ ,  $\hat{\rho}^{\text{out}}$  is also simply written in the form  $p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|$ , and we see that  $\|\hat{\rho}^{\text{in}}\| > \|\hat{\rho}^{\text{out}}\|$  holds. When  $\alpha_2 > 0$ , we can easily confirm that all of the nine matrix elements  $\langle i|\hat{\rho}^{\text{out}}|j\rangle$  are strictly positive. Then, there is a state  $|\xi\rangle := \sum_j c_j |j\rangle$  with  $c_j > 0$  such that  $\|\hat{\rho}^{\text{out}}\| = \langle \xi|\hat{\rho}^{\text{out}}|\xi\rangle$  (Perron's theorem [16]). This leads to  $\|\hat{\rho}^{\text{in}}\| \geq 2\kappa c_0 c_1 + \|\hat{\rho}^{\text{out}}\| > \|\hat{\rho}^{\text{out}}\|$ .

In order to prove Eq. (23), we consider  $\hat{\rho}' := \hat{P}\hat{\rho}^{\text{in}}\hat{P}$ , where  $\hat{P} := |0\rangle_A\langle 0| + |1\rangle_A\langle 1|$ . Since  $\hat{\rho}'$  is represented by a  $2 \times 2$  matrix, it is tedious but not difficult to show that  $\det[\hat{\rho}' - (1/2)\hat{P}] < 0$  holds for  $\alpha_0 > 0, \alpha_1 > 0$  [17]. This gives  $E_{A;BB''}^{(2)}(\Psi^{\text{in}}) = 1 - \|\hat{\rho}^{\text{in}}\| \leq 1 - \|\hat{\rho}'\| < 1/2$ . Combining this with  $E_{AA';B''}^{(2)}(\Psi_{AA'}^{\text{out}}) = 1/2$ , we obtain Eq. (23).

Having proved Eqs. (22) and (23), the monotonicity Eq. (16) leads to

$$E_{A;BB''}^{(2)}(\Psi^{\text{in}}) \geq \gamma_{AA'} E_{AA';B''}^{(2)}(\Psi_{AA'}^{\text{out}}) + \gamma_{AB} E_{A;BB''}^{(2)}(\Psi_{AB}^{\text{out}}) \quad (24)$$

$$> (\gamma_{AA'} + \gamma_{AB}) E_{A;BB''}^{(2)}(\Psi^{\text{in}}), \quad (25)$$

and hence

$$\gamma_{AA'} + \gamma_{AB} < 1. \quad (26)$$

From Eqs. (19) and (26), Lemma 1 is proved.

A few remarks may be worth mentioning about Lemma 1. For simplicity, let us consider a deterministic transformation  $|\psi_i\rangle_A |\phi_i\rangle_B \rightarrow |\psi_i\rangle_A |\psi_i\rangle_B$ . Since  $\langle \phi_1 | \phi_2 \rangle > \langle \psi_1 | \psi_2 \rangle (= 0)$ , distinguishability in system  $B$  alone between state 1 and 2 is improving in the process, implying that the process may be regarded as a kind of measurement on system  $A$  that tries to distinguish between  $|0\rangle_A$

and  $|1\rangle_A$ . One might expect that in such a case, any superposition state between  $|0\rangle_A$  and  $|1\rangle_A$  will be destroyed in the process. That would be surely true if the initial state of system  $B$ , which we may regard as the measurement apparatus, was independent of  $i$ . But the present case corresponds to an atypical measurement in which the initial state of system  $B$  depends on  $i$ . Then, rather surprisingly, there is a special initial state,  $|\phi_3\rangle_B$ , that enables the process to keep a superposition state,  $|\psi_3\rangle_A$ , unaltered. Lemma 1 means that this strange process, extracting information while retaining a superposition, can only be realized by interacting systems  $A$  and  $B$  in a purely quantum way.

Finally, we show that we can *always* find such a subtle way of giving supplementary information when the set of original states is irreducible but not PNO.

*Theorem 1* — For any non-PNO irreducible set  $\{|\psi_i\rangle_A\}_{i=1,\dots,n}$ , there exists a set of supplementary states  $\{|\phi_i\rangle_B\}_{i=1,\dots,n}$  such that Alice and Bob can never achieve the cloning task of Eq. (5) over LOCC.

*Proof.* Let us call sequence  $|\Gamma_1\rangle, \dots, |\Gamma_m\rangle$  a “chain” if  $\langle\Gamma_i|\Gamma_{i+1}\rangle \neq 0$  for  $i = 1, \dots, m-1$ . First, we show that  $S := \{|\psi_i\rangle_A\}$  includes a chain  $|\psi_{i_0}\rangle, |\psi_{i_1}\rangle, |\psi_{i_2}\rangle$  with  $\langle\psi_{i_0}|\psi_{i_2}\rangle = 0$ . Let  $|\xi\rangle$  and  $|\zeta\rangle$  be a pair of orthogonal states in the non-PNO set  $S$ . Since  $S$  is irreducible, it includes a chain  $|\xi\rangle, |\eta_1\rangle, |\eta_2\rangle, \dots, |\eta_m\rangle, |\zeta\rangle$  of length  $m+2$  ( $m \geq 1$ ). If  $m = 1$ , this is the chain we seek. If  $m \geq 2$  and  $\langle\xi|\eta_2\rangle = 0$ , we obtain the desired chain  $|\xi\rangle, |\eta_1\rangle, |\eta_2\rangle$ . When  $\langle\xi|\eta_2\rangle \neq 0$ , we can remove  $|\eta_1\rangle$  from the chain and the remaining sequence of length  $m+1$  still forms a chain connecting  $|\xi\rangle$  and  $|\zeta\rangle$ . Hence, repeating the procedure, we can always find a chain  $|\psi_{i_0}\rangle, |\psi_{i_1}\rangle, |\psi_{i_2}\rangle$  with  $\langle\psi_{i_0}|\psi_{i_2}\rangle = 0$ . Let us relabel the index  $i$  in  $S$  such that this chain becomes  $|\psi_1\rangle, |\psi_3\rangle, |\psi_2\rangle$ . If we choose an appropriate basis, these states are written as in Eqs. (6)-(8). If we define the supplementary states simply by  $|\phi_i\rangle$  of Eqs. (9)-(11) for  $i = 1, 2, 3$  and  $|\phi_i\rangle := |i-1\rangle$  for  $i = 4, \dots, n$ , the task of cloning becomes equivalent to the case with the three states considered in Lemma 1, and hence Theorem 1 is proved.

The present results, combined with the prior knowledge, reveal the general property of quantum information in a set  $S := \{|\psi_i\rangle\}$  that manifests when one tries to clone it. For the simple cloning, what matters is the reducibility of the set  $S$ . The reducible part is purely classical, which is freely cloned. The irreducible part cannot be cloned at all, which represents a quantum nature. If one has an additional system with supplementary information  $\{\hat{\rho}_i\}$ , which has partial but not enough information to produce a copy  $|\psi_i\rangle$  on its own, the class of irreducible sets are further divided into two types showing quite opposite behavior: When the set  $S$  is PNO, the original system is not helpful at all, and the cloning is still forbidden, which is the stronger no-cloning theorem [12]. When the set  $S$  is not PNO (but irreducible), the original sys-

tem can help to achieve the cloning — this fact itself is not surprising, since one may interpret that the orthogonal pairs of states in  $S$  hold information in just a classical way. What is surprising is that we can always find an example of  $\{\hat{\rho}_i\}$  such that this help is available only through a purely quantum operation that is not achieved over LOCC. Hence the two cases, PNO and non-PNO, have properties which are both purely quantum but are in a striking contrast with each other.

We have seen that the supplementary-state scenario is very helpful in grasping the nature of quantum information in a qualitative way. The scenario has also been combined with other protocols such as probabilistic cloning [18] and a novel cloning machine [19, 20]. We believe that we may also obtain a detailed quantitative understanding by combining it with more elaborate protocols.

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